A GEOMETRIC CRITERION FOR SYZYGIES IN EQUIVARIANT COHOMOLOGY

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Abstract. Let T be a torus and X an orientable T-manifold such that the action is almost free outside of finitely many submanifolds of codimension 2. We characterize geometrically when the equivariant cohomology $H_T^*(X)$ is a certain syzygy as module over $H^*(BT)$. It turns out that this only depends on the "face structure" of the orbit space X/T. Our criterion unifies and generalizes results of many authors about the freeness and torsion-freeness of equivariant cohomology for various classes of T-manifolds. It also permits to easily construct new examples.

1. Introduction

Let R be a polynomial ring in r variables over a field. Recall that a finitely generated R-module M is a j-th syzygy if there is an exact sequence

$$0 \to M \to F^1 \to \cdots \to F^j \tag{1.1}$$

with finitely generated free R-modules $F^1, \ldots, F^j$. The first syzygies are exactly the torsion-free modules, and the r-th syzygies the free modules. Syzygies therefore interpolate between torsion-freeness and freeness. In [1], Allday, Puppe and the author initiated the study of syzygies in the context of equivariant cohomology.

More precisely, let $T \cong (S^1)^r$ be a torus and $R = H^*(BT)$ the cohomology of its classifying space with rational coefficients. Consider the orbit filtration

$$X^T = X_0 \subset X_1 \subset \cdots \subset X_r = X \tag{1.2}$$

of a "nice" T-space X, where the equivariant i-skeleton X_i is the union of the T-orbits of dimension at most i. It leads to an "Atiyah–Bredon sequence"

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to \cdots \to H_T^{*+r}(X_r, X_{r-1}) \to 0 \tag{1.3}$$

in equivariant cohomology. We call $H_T^{*+i}(X_i, X_{i-1})$ the i-position of this sequence; $H_T^*(X)$ is at position -1. One of the main results of [1] is as follows:

Theorem 1.1 ([1, Thm. 5.6]). *The Atiyah–Bredon sequence* (1.3) *is exact at all positions $i \le j-2$ if and only if $H_T^*(X)$ is a j-th syzygy.*

Theorem 1.1 interpolates between the classical statement that the restriction map $H_T^*(X) \to H_T^*(X_0)$ is injective if and only if $H_T^*(X)$ is torsion-free and a result of Atiyah and Bredon [6, Main Lemma] saying that the whole sequence (1.3) is exact if $H_T^*(X)$ is free.

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Equally important is the case $j = 2$, which states that the "Chang–Skjelbred sequence"

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \tag{1.4}$$

is exact if and only if $H_T^*(X)$ is a reflexive R-module (a second syzygy). In this case one can efficiently compute $H_T^*(X)$ out of data related only to the fixed points and the 1-dimensional orbits ("GKM method"). It turns out that for a Poincaré duality space X the second syzygies also characterize the perfection of the equivariant Poincaré pairing, see [1, Prop. 5.8].

One would therefore like to to have an easy criterion to decide whether or not $H_T^*(X)$ is a certain syzygy. In this paper we provide such a criterion for a large class of T-spaces X. Specifically, we assume the following:

(1) X is an orientable connected smooth T-manifold,
(2) $H^*(X)$ is finite-dimensional, and
(3) X_{r-1}, the complement of the full-dimensional orbits, is the union of finitely many closed connected submanifolds of codimension 2.

Smooth toric varieties and, more generally, torus manifolds [16] and open torus manifolds [15] are of this form. Our class also includes interesting T-manifolds whose fixed point set is not discrete, for example the "mutants" constructed in [11].

Let X be a T-manifold satisfying our assumptions. We call the closure P of a connected component of $(X_i \setminus X_{i-1})/T$ a face of X/T of rank i. If all isotropy groups are connected, then the orbit space is a manifold with corners, and P is a face of X/T in the usual sense. Filtering P according to the rank of its faces leads to a complex $B_c^*(P)$ with

$$B_c^i(P) = \bigoplus_{\substack{Q \subset P \\ \operatorname{rank} Q = i}} H_c^{*+i}(Q, \partial Q), \tag{1.5}$$

where $H_c^*(-)$ denotes rational cohomology with compact supports.

Our main result shows that whether or not $H_T^*(X)$ is a certain syzygy only depends on the face structure of the orbit space:

Theorem 1.2. *Let X be a T-manifold satisfying the assumptions above. Then $H_T^*(X)$ is a j-th syzygy if and only if $H^i(B_c^*(P)) = 0$ for all faces P of X/T and all $i > \max(\operatorname{rank} P - j, 0)$.*

This characterization unifies and extends results of many authors concerning the freeness and torsion-freeness of torus-equivariant cohomology. This includes work of Barthel–Brasselet–Fieseler–Kaup [3], Masuda–Panov [16], Masuda [15] and Goertsches–Rollenske [12]. We also generalize a result of Bredon [6] about the cohomology of X/T, and we recover the calculation of Ext modules of Stanley–Reisner rings done by Mustaţă [17] and Yanagawa [18].

Compact orientable T-manifolds of dimension $2r$ have received a lot of interest so far, for example complete toric varieties or torus manifolds. We can now explain why these spaces do not provide interesting examples of syzygies.

Corollary 1.3. *Let X be a compact T-manifold of dimension $2r$ satisfying the assumptions above. Then $H_T^*(X)$ is torsion-free if and only if it is free over R.*

The paper is organized as follows: In Section 2 we characterize syzygies by a depth condition on the equivariant cohomology of fixed point sets of subtori.

Theorem 1.2 is proved in Section 4, based on a certain decomposition of the Atiyah–Bredon sequence for compact supports that we establish in Section 3. We comment in Section 5 on the relation to previous work; in Section 6 we finally illustrate the computational power of Theorem 1.2 by revisiting and generalizing the examples that were considered in [1].

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2. Syzygies and fixed point sets

We work over a field $\Bbbk$ of characteristic 0; all tensor products are over $\Bbbk$ unless indicated otherwise. Also, all cohomology is taken with coefficients in $\Bbbk$. In particular, $R = H^*(BT)$ is a polynomial ring in r generators of degree 2 with coefficients in $\Bbbk$. Here and elsewhere $H^*(-)$ denotes singular or, equivalently, Alexander–Spanier cohomology.

From the next section on, we will only consider smooth T-manifolds. However, in this section we more generally allow the same T-spaces X as in [1]. This means that X must be a locally compact, second-countable Hausdorff space such that $H^*(X)$ is a finite-dimensional $\Bbbk$-vector space, and each X_i must be locally contractible. Also, only finitely many subtori of T can occur as the identity component of an isotropy group in X.

We write $\overline{AB}^*(X)$ or, if necessary, $\overline{AB}^*_T(X)$ for the (augmented) Atiyah–Bredon sequence (1.3), and $AB^*(X)$ for the non-augmented sequence obtained by removing the leading term $\overline{AB}^{-1}(X) = H^*_T(X)$. The analogous sequences for equivariant cohomology with compact supports are denoted by $\overline{AB}^*_c(X)$ and $AB^*_c(X)$.

A good reference for syzygies is [8, Sec. 16E]; see also the very short summary in [1, Sec. 2.3]. Recall that the depth of a finitely generated R-module M is the maximal length of an M-regular sequence; it is related to the homological dimension of M by the formula

$$\operatorname{depth} M + \operatorname{hd} M = r. \tag{2.1}$$

Proposition 2.1. *$H^*_T(X)$ is a j-th syzygy if and only if*

$$\operatorname{depth}_{H^*(BL)} H^*_L(X^K) \geq \min(j, \dim L)$$

for any subtorus $K \subset T$ with quotient $L = T/K$.

Proof. By [8, Prop. 16.29], $H^*_T(X)$ is a j-th syzygy if and only if

$$\operatorname{depth}_{R_\mathfrak{p}} H^*_T(X)_\mathfrak{p} \geq \min(j, \operatorname{depth} R_\mathfrak{p}) \tag{2.2}$$

holds for all prime ideals $\mathfrak{p} \lhd R$.

Fix a prime ideal $\mathfrak{p} \lhd R$ and let $\mathfrak{q} \subset \mathfrak{p}$ be the prime ideal generated by $\mathfrak{p} \cap H^2(BT; \mathbb{Q})$. Then $\mathfrak{q}$ is the kernel of the restriction map $H^*(BT) \to H^*(BK)$ for some subtorus $K \subset T$ with quotient $L = T/K$, which we may identify with a torus complement to K in T. We set $R' = H^*(BK)$ and $R'' = H^*(BL)$. By the localization theorem [2, Ex. 3.1.5, Thm. 3.2.6], we have

$$H^*_T(X)_\mathfrak{p} = H^*_T(X^K)_\mathfrak{p} = \bigl(R \otimes_{R''} H^*_L(X^K)\bigr)_\mathfrak{p} = R_\mathfrak{p} \otimes_{R''} H^*_L(X^K). \tag{2.3}$$

We observe that $R_{\mathfrak{p}}$ is a faithfully flat R''-module. This is because $R \cong R' \otimes R''$ is a free R''-module, localization is exact and the localization map

$$R \otimes_{R''} N \cong R' \otimes N \to (R' \otimes N)_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_{R''} N \tag{2.4}$$

is injective for any R''-module N, *cf.* [9, Lemma 1.2]. (Write an $a \notin \mathfrak{p}$ as a linear combination of elements of the form $a' \otimes a'' \in R' \otimes R''$. Then it must have a component in $R' \otimes \Bbbk$, so that multiplication by a is injective on $R' \otimes N$.)

Now for any R''-module N we have

$$\operatorname{Ext}^*_R(R \otimes_{R''} N, R) = R \otimes_{R''} \operatorname{Ext}^*_{R''}(N, R''), \tag{2.5}$$

hence

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(H^*_T(X)_{\mathfrak{p}}, R_{\mathfrak{p}}) = R_{\mathfrak{p}} \otimes_{R''} \operatorname{Ext}^i_{R''}(H^*_L(X), R'') \tag{2.6}$$

vanishes if and only if $\operatorname{Ext}^i_{R''}(H^*_L(X), R'')$ does. Since $\dim R'' \le \dim R_{\mathfrak{p}}$, the characterization of depth via Ext [7, Cor. 3.5.11] finishes the proof. □

Corollary 2.2. *If $H^*_T(X)$ is a j-th syzygy over R, then so is $H^*_L(X^K)$ over $H^*(BL)$ for any subtorus $K \subset T$ with quotient $L = T/K$.*

Corollary 2.3.

1. *Assume that X is a Poincaré duality space over $\Bbbk$. Then $H^*_T(X)$ is a j-th syzygy if and only if $H^i(AB^*_L(X^K)) = 0$ for any subtorus $K \subset T$ with quotient $L = T/K$ and any $i > \max(\dim L - j, 0)$.*
2. *Assume that X is an orientable $\Bbbk$-homology manifold. Then $H^*_T(X)$ is a j-th syzygy if and only if $H^i(AB^*_{L,c}(X^K)) = 0$ for any subtorus $K \subset T$ with quotient $L = T/K$ and any $i > \max(\dim L - j, 0)$.*

Proof. We start by recalling another result from [1]: The cohomology of the non-augmented Atiyah–Bredon sequence $AB^*(X)$ can be expressed as an Ext module involving the (suitably defined) equivariant homology of X,

$$H^i(AB^*(X)) = \operatorname{Ext}^i_R(H^T_*(X), R) \tag{2.7}$$

for any $i \ge 0$, see [1, Thm. 5.1].

By Proposition 2.1, $H^*_T(X)$ is a j-th syzygy if and only if

$$\operatorname{depth}_{H^*(BL)} H^*_L(X^K) \ge \min(j, \dim L) \tag{2.8}$$

for any K. If X is a PD space, then so is each connected component of X^K, *cf.* [2, Thm. 5.2.1, Rem. 5.2.4]. By equivariant Poincaré duality [1, Prop. 3.5] we have $H^*_L(X^K) \cong H^L_*(X^K)$, up to a degree shift. Applying (2.7) to the L-space X^K one therefore gets

$$H^i(AB^*_L(X^K)) = \operatorname{Ext}^i_{H^*(BL)}(H^L_*(X^K), H^*(BL)), \tag{2.9}$$

and the claim follows from characterization of depth via Ext.

The second part is proved analogously. One uses the fact that each connected component of X^K is again an orientable homology manifold, *cf.* [5, Thm. V.3.2], and versions of equivariant Poincaré duality and identity (2.7) for (co)homology with appropriate supports, see [1, Prop. 6.1, Thm. 6.6]. □

3. A decomposition of the Atiyah–Bredon sequence

For the rest of this paper we impose the following restrictions on X:

(1) X is an orientable connected smooth T-manifold,
(2) $H^*(X)$ is finite-dimensional, and
(3) X_{r-1}, the complement of the full-dimensional orbits, is the union of finitely many closed connected submanifolds $Z_1, \ldots, Z_k$ of codimension 2. Following [16] and [15], we call them the *characteristic submanifolds* of X.

Smooth toric varieties and, more generally, torus manifolds [16] and open torus manifolds [15] are of this form, as are the "mutants" constructed in [11]. These classes of spaces will be discussed in more detail in Section 5.

Let $\pi\colon X \to X/T$ be the quotient by the T-action. We identify X^T with its image in X/T. The closure P of a connected component $\dot{P}$ of $\pi(X_i \setminus X_{i-1}) \subset X/T$ is called a *face* of X/T of *rank* i, with interior $\dot{P}$ and boundary $\partial P = P \setminus \dot{P}$. The faces of X/T form a poset $\mathcal{P}$ under inclusion. If $P < Q$ and $\operatorname{rank} P = \operatorname{rank} Q - 1$, we say that P is a *facet* of Q, in symbols $P <_1 Q$.

For a face $P \in \mathcal{P}$ we define the invariant submanifolds

$$X^{\dot{P}} = \pi^{-1}(\dot{P}) \subset X^P = \pi^{-1}(P) \subset X, \tag{3.1}$$

and we write $T_P \subset T$ for the subtorus that is the common identity component of the isotropy groups of the points in $X^{\dot{P}}$. We also define the algebra $R_P = H^*(BT_P)$ as well as restriction maps $\rho_P\colon R \to R_P$ and $\rho_{PQ}\colon R_P \to R_Q$ for $P \le Q$.

Let $P \in \mathcal{P}$. Any choice of decomposition $T = T_P \times L$ with $L \cong T/T_P$ leads to an isomorphism of algebras

$$\varphi_{P,L}\colon H^*(BL) \otimes R_P \to R \tag{3.2}$$

as well as to an isomorphism of R-algebras

$$\psi_{P,L}\colon H_c^*(\dot{P}) \otimes R_P \to H_{T,c}^*(X^{\dot{P}}), \tag{3.3}$$

where the R-module structure on the left hand side is defined via $\psi_{P,L}$ and the isomorphism $H_c^*(\dot{P}) = H_{L,c}^*(X^{\dot{P}})$. We define an increasing filtration on $H_{T,c}^*(X^{\dot{P}})$ by

$$\mathcal{F}_j\, H_{T,c}^*(X^{\dot{P}}) = \psi_{P,L}\big(H_c^*(\dot{P}) \otimes H^{\le j}(BT_P)\big). \tag{3.4}$$

for $j \ge 0$.

Lemma 3.1. *This filtration is independent of the choice of L and thus induces a canonical isomorphism of R-algebras*

$$\psi_P\colon H_c^*(\dot{P}) \otimes R_P \to \operatorname{Gr} H_{T,c}^*(X^{\dot{P}}),$$

where the R-module structure on the left is defined via the restriction $R \to R_P$.

Proof. If $T = T_P \times L'$ is a different decomposition and $t \in R_P$ of degree 2, then $\psi_{P,L'}(1 \otimes t) = \psi_{P,L}(1 \otimes t + \alpha \otimes 1)$ for some $\alpha \in H_c^2(\dot{P})$. This proves that ψ_P does not depend on the chosen decomposition. The claim about the R-module structure follows because elements of positive degree in $H^*(BL) \subset R$ do not increase the filtration degree. ∎

Analogously to (3.4), we obtain a canonical filtration of the Atiyah–Bredon sequence $AB_c^*(X)$ with

$$\mathcal{F}_j\, AB_c^i(X) = \bigoplus_{\operatorname{rank} P = i} \mathcal{F}_j\, H_{T,c}^*(X^{\dot P}) \tag{3.5}$$

and an isomorphism of R-algebras

$$\operatorname{Gr} AB_c^i(X) = \bigoplus_{\operatorname{rank} P = i} H_c^*(\dot P) \otimes R_P. \tag{3.6}$$

The left hand side inherits a differential from $AB_c^*(X)$. Our goal is to relate this complex to the orbit space X/T.

The stratification of a face P of X/T by its faces leads to a spectral sequence for cohomology with compact supports. The i-th column of the E_1 page is

$$B_c^i(P) := \bigoplus_{\substack{Q \le P \\ \operatorname{rank} Q = i}} H_c^{*+i}(\dot Q). \tag{3.7}$$

We also need "good" generators of the algebras R_P. It is here that the characteristic submanifolds come into play. Notice that their normal bundles are complex line bundles.

Lemma 3.2. *Let $P \in \mathcal{P}$ be of rank p. Then X^P is the intersection of exactly $r - p$ characteristic submanifolds Z_i. Moreover, the normal bundle of X^P is the direct sum of the restrictions of the normal bundles of these Z_i.*

Proof. Take an $x \in X^{\dot P}$. By the differentiable slice theorem, there is a T-stable neighbourhood U of Tx and a T_P-module V such that U is equivariantly diffeomorphic to a T-stable neighbourhood W of $[1, 0]$ in $T \times_{T_P} V$. Let $\chi_1, \ldots, \chi_i \in \mathfrak{X}(T_P)$ be the non-zero characters occurring in V, with multiplicities. It suffices to prove that they form a basis of $\mathfrak{X}(T_P) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Since T acts almost freely on a dense subset of X, the same applies to T_P and V. Hence the χ_j's span $\mathfrak{X}(T_P) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now assume that they are not linearly independent. By replacing P by one of its faces, we may assume that any non-trivial relation among the $\chi_j's$ involves all of them. But this means that V^K has (real) codimension at least 4 for any circle $K \subset T_P$, and the same holds for the component of $X^K \subset X$ containing x. In other words, x does not lie in any characteristic submanifold, which is impossible. Hence the non-zero characters are linearly independent. □

Remark 3.3. Assume that all isotropy groups in X are connected. It then follows from Lemma 3.2 that the orbit space X/T is a manifold with corners[1], whose faces of codimension j are the faces of X/T of rank $r - j$ introduced above.

Assume additionally that any face of X/T contains a component of X^T; we will see in Theorem 4.4 that this is the interesting case for our purposes. Write $m = \dim X - 2r$ for the dimension of X^T. As in the proof of Lemma 3.2, the differentiable slice theorem then implies that locally the T-action on X looks like the standard action of $(S^1)^r$ on $\mathbb{C}^r \times \mathbb{R}^m$ with trivial action on the second factor. For $m = 0$, these are the "locally standard actions" considered in [16] and [15].

[1]For the definition of a manifold with corners, see for instance [13, Sec. 1.1]. We assume as part of the definition that any face of codimension j is contained in exactly j facets. This is called a "manifold with faces" in [13].

Let $P \in \mathcal{P}$ and consider the normal bundle to X^P. By Lemma 3.2 it is the direct sum of the restrictions of the line bundles corresponding to the facets of X/T containing P. There is an obvious subbundle for any face $O \geq P$, and by restricting further to any $x \in X^P$ we get an equivariant Euler class $t_O^P \in R_P$, which is the product of the t_F^P with $O \leq F <_1 X/T$. In particular, the t_F^P with $P \leq F <_1 X/T$ are canonical generators of R_P. By naturality, these classes satisfy

$$\rho_{PQ}\big(t_O^P\big) = \begin{cases} t_O^Q & \text{if } O \leq Q, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

To simplify notation, we write $t_O = t_O^P$. Note that the R_P-module $R_P\, t_P$ is isomorphic to R_P itself with a degree shift by $2 \operatorname{codim} P$.

Proposition 3.4. *For any $i \geq 0$, there is an isomorphism of graded vector spaces*

$$\Phi_X \colon H^i\big(\operatorname{Gr} AB_c^*(X)\big) = \bigoplus_{P \in \mathcal{P}} H^i(B_c^*(P)) \otimes R_P\, t_P.$$

Proof. By collecting monomials in same generators, we arrive at isomorphisms of graded vector spaces

$$R_P = \bigoplus_{P \leq Q} R_Q\, t_Q \tag{3.9}$$

and

$$\operatorname{Gr} AB_c^*(X) = \bigoplus_{P \in \mathcal{P}} B_c^*(P) \otimes R_P\, t_P. \tag{3.10}$$

We have to show that (3.10) is compatible with the differentials. By equation (3.8), it suffices to show that for any $P <_1 Q$ the diagram

$$\begin{tikzcd} H_c^*(\dot P) \otimes R_P \arrow[r, "\delta \otimes \rho_{PQ}"] \arrow[d, "\psi_P"'] & H_c^{*+1}(\dot Q) \otimes R_Q \arrow[d, "\psi_Q"] \\ \operatorname{Gr} H_{T,c}^*(X^{\dot P}) \arrow[r, "\delta"] & \operatorname{Gr} H_{T,c}^{*+1}(X^{\dot Q}) \end{tikzcd} \tag{3.11}$$

is well-defined and commutes. Here the pair $(\dot P \cup \dot Q, \dot P)$ gives rise to the boundary map δ in the top row, and the inverse image of this pair under π to the boundary map in the bottom row. Since the boundary map in equivariant cohomology is R-linear, the claim follows from the R-module structure as given by Lemma 3.1. $\square$

The non-augmented Atiyah–Bredon sequence $AB_c^*(X)$ is the E_2 page of a spectral sequence of algebras converging to $H_{T,c}^*(X)$, hence is equipped with a multiplication induced by the cup product. The same applies to the associated graded complex $\operatorname{Gr} AB_c^*(X)$. We can describe the latter multiplication as well as the module structure over R in terms of the isomorphism Φ_X established above. This is similar to the multiplicative decomposition obtained by Bifet–De Concini–Procesi for the equivariant cohomology of regular embeddings [4, Prop. 16].

Like $AB_c^*(X)$, each complex $B_c^*(P)$ has a product, which is natural with respect to the inclusion $O \hookrightarrow P$ of faces $O \leq P$. For an element $\alpha \in B_c^*(P)$, we write $\alpha|_O$ for its restriction to $B_c^*(O)$, and similarly in cohomology.

Note that for any pair of faces P, Q there is a minimal face containing both P and Q; it is denoted by $P \vee Q$. Also note that the intersection $P \cap Q$ is (empty or) the disjoint union of faces O of codimension $\operatorname{codim} P + \operatorname{codim} Q$. Following [16], we write $O \in P \cap Q$ for these faces.

Proposition 3.5. *Let P, $Q \in \mathcal{P}$, $\alpha \otimes f\, t_P \in H^*(B_c^*(P)) \otimes R_P\, t_P$ and $\beta \otimes g\, t_Q \in H^*(B_c^*(Q)) \otimes R_Q\, t_Q$. The product of these elements under the isomorphism Φ_X is*

$$(\alpha \otimes f\, t_P) \cdot (\beta \otimes g\, t_Q) = \sum_{O \in P \cap Q} \alpha|_O \cup \beta|_O \otimes f g\, t_P t_Q.$$

Moreover, a linear element $t \in R$ acts by multiplication with the element

$$\sum_{F <_1 X/T} 1 \otimes \rho_F(t).$$

Note that $f g\, t_P t_Q$ lies in $R_O\, t_O$ for $O \in P \cap Q$ because t_F lies in R_O for every facet F of X/T containing P or Q and because $t_P t_Q$ is divisible by t_O. Moreover, if $t \in R$ is linear, then the restriction $\rho_F(t) \in R_F$ is a multiple of t_F.

Proof. For the first claim, it is enough to verify the analogous identity for the underlying differential graded algebra, where we can do it componentwise for each summand $H_c^{*+k}(\dot{O}') \otimes R_{O'} \subset \operatorname{Gr} AB_c^*(X)$ corresponding to a face $O' \in \mathcal{P}$ of rank k.

Assume first that $P = Q = X/T$. In this case the claimed identity follows from the multiplicativity of the map $B_c^*(X/T) \to \operatorname{Gr} AB_c^*(X)$, which is an isomorphism onto the graded piece corresponding to the lowest filtration degree.

For the general case we use that the multiplication in $\operatorname{Gr} AB_c^*(X)$ is R-bilinear. Since the product $t_P t_Q$ restricts to $0 \in R_{O'}$ if O' is not a face of some $O \in R \cap Q$, the component corresponding to such an O' vanishes as claimed. On the other hand, for a face $O' \le O \in P \cap Q$ the identity holds because the restriction map $B_c^*(X/T) \to B_c^*(O)$ is multiplicative.

The R-module structure also follows from the R-bilinearity of the product. □

Remark 3.6. Assume that for any $P \in \mathcal{P}$ the graded vector space $H_c^*(\dot{P})$ is concentrated in at most one single even degree and one single odd degree. By inspecting the proof of Lemma 3.1, one sees that the isomorphism (3.3) is canonical in this case. As a consequence, Propositions 3.4 and 3.5 hold true without passing to the associated graded object. That is,

$$H^i\big(AB_c^*(X)\big) = \bigoplus_{P \in \mathcal{P}} H^i(B_c^*(P)) \otimes R_P\, t_P \tag{3.12}$$

with the same R-algebra structure as in Proposition 3.5.

Example 3.7. Consider $X = \mathbb{CP}^1 = S^2$ with the usual rotation action of the circle. Then $R = \Bbbk[t]$, and X/T is an interval with vertices P and Q. Hence

$$H^0\big(\operatorname{Gr} AB_c^*(X)\big) \cong R\, t_P \oplus R\, t_Q \oplus \Bbbk \tag{3.13}$$

and $H^i(\operatorname{Gr} AB_c^*(X)) = 0$ for $i > 0$. The product of t_P and t_Q vanishes as P and Q do not intersect. Using Remark 3.6, we can recover the familiar presentation of the algebra $H_T^*(\mathbb{CP}^1)$ as a Stanley–Reisner algebra,

$$H_T^*(\mathbb{CP}^1) \cong \Bbbk[t_P, t_Q] \;/\; (t_P t_Q). \tag{3.14}$$

4. A geometric characterization of syzygies

Proposition 3.4 will prove one direction of Theorem 1.2. For the other one we need to blow-up the characteristic submanifolds of X. The most convenient way to do so is to replace each characteristic submanifold Z_i by the sphere bundle of its normal bundle. This leads to a T-manifold with corners Y. See [13, Sec. 1.1 & 1.3] for the definition as well as for those of an invariant submanifold Z of Y and the blow-up along Z. Note that a submanifold of a manifold with corners is assumed to be transverse to all faces. For the following construction, compare also the proof of [10, Prop. 1.4].

Lemma 4.1. *There is a T-manifold with corners Y such that $Y_{r-1} = \emptyset$ and an equivariant map $\beta\colon Y \to X$ inducing an isomorphism $Y/T \to X/T$.*

Proof. By induction, it suffices to prove the following claim: Let X be a T-manifold with corners, and let $Z_1, \ldots, Z_k$, $k \geq 1$, be the characteristic submanifolds of X (which are invariant submanifolds in the sense described above). Then there is a T-manifold with corners Y and an equivariant map $\beta\colon Y \to X$ inducing an isomorphism $Y/T \to X/T$. Moreover, the characteristic submanifolds of Y are $\beta^{-1}(Z_1), \ldots, \beta^{-1}(Z_{k-1})$.

Let $K \subset T$ be the 1-dimensional subgroup fixing $Z = Z_k$, and let $\beta\colon Y \to X$ be the blow-up along Z. Since the codimension of Z is 2, this sphere bundle is a circle bundle, and the fibres are orbits of K. Hence, $\beta\colon \beta^{-1}(Z) \to Z$ is the quotient by K, which implies that $Y/T \to X/T$ is an isomorphism.

Observe that the sphere bundle intersects all faces of X and the other characteristic submanifolds transversally. Since $\dim Ty = \dim T\beta(y) + 1$ for any $y \in \beta^{-1}(Z)$, the claim about the characteristic submanifolds follows. □

Example 4.2. Let X be a smooth projective toric variety or a "toric manifold" in the sense of [10]. Then X can be reconstructed from the simple polytope $X/T = Q$ as the identification space

$$X = (T \times Q) \,/ \sim, \tag{4.1}$$

where the equivalence relation identifies T_P-orbits over the interior of each face P, *cf.* [10, Prop. 1.4]. The blow-up of X constructed in Lemma 4.1 is nothing but the quotient map

$$\beta\colon T \times Q \to (T \times Q) \,/ \sim. \tag{4.2}$$

Remark 4.3. We observed in Remark 3.3 that X/T is a manifold with corners if all isotropy groups are connected. In general, it is an "orbifold with corners". Since we are using coefficients in a field of characteristic 0, the quotient still behaves homologically like a manifold with corners. The easiest way to see this is to use the isomorphism

$$H^*(S) = H_T^*(\beta^{-1}(\pi^{-1}(S))) \tag{4.3}$$

for any subset $S \subset X/T$ and its relatives for equivariant homology and other supports, combined with equivariant Poincaré duality for Y, *cf.* [1, Sec. 3 & 6].

We can now prove our main result.

Theorem 4.4. *$H_T^*(X)$ is a j-th syzygy if and only if $H^i(B_c^*(P)) = 0$ for all $P \in \mathcal{P}$ and all $i > \max(\operatorname{rank} P - j, 0)$.*

Proof. It follows from Lemma 3.2 that X satisfies the requirement on isotropy groups imposed in Section 2. By Corollary 2.3 (2), it therefore suffices to show that the condition above,

$$\text{(A)} \qquad H^i(B_c^*(P)) = 0 \quad \forall\ P \in \mathcal{P},\ i > \max(\operatorname{rank} P - j, 0)$$

is equivalent to

$$\text{(B)} \quad H^i(AB_{L,c}^*(X^K)) = 0 \quad \forall \text{ subtori } K \subset T,\ L = T/K,\ i > \max(\dim L - j, 0).$$

(A) $\Rightarrow$ (B): The connected components of X^K are of the form X^Q for certain faces Q. We consider these components separately. For the moment, we also assume $K = T_Q$, so that $\dim L = \operatorname{rank} Q$. By Proposition 3.4 and condition (A) we have

$$H^i\bigl(\operatorname{Gr} AB_{L,c}^*(X^Q)\bigr) = \bigoplus_{P \le Q} H^i(B_c^*(P)) \otimes R_P\, t_P = 0 \tag{4.4}$$

for all $i > \max(\dim L - j, 0)$ since $\operatorname{rank} P \le \operatorname{rank} Q$ for any $P \le Q$. (Here $R_P\, t_P$ has to be understood with respect to the L-space X^Q.) Because (4.4) is the E_1 page of a spectral sequence converging to $H^*(AB_{L,c}^*(X^Q))$, the claim follows.

If $K \subsetneq T_Q$, then

$$H^i\bigl(\operatorname{Gr} AB_{L,c}^*(X^Q)\bigr) = H^i\bigl(\operatorname{Gr} AB_{L',c}^*(X^Q)\bigr) \otimes H^*(BK') \tag{4.5}$$

where $K' = T_Q/K$ and $L' = T/T_Q$, which reduces this case to the previous one.

$\neg$(A) $\Rightarrow$ $\neg$(B): Choose a $P \in \mathcal{P}$ and an $i > \max(\operatorname{rank} P - j, 0)$ such that $H^i(B_c^*(P)) \ne 0$. Let $K = T_P$, so that $X^P =: X'$ is a connected component of X^K. Note that X' together with the action of $L = T/K$ satisfies the assumptions on spaces stated at the beginning of this section. Let $\beta\colon Y' \to X'$ be the blow-up obtained in Lemma 4.1. Filtering Y' by the inverse image of the orbit filtration of X' leads to a spectral sequence, and the map

$$B_c^j(P) \to AB_{L,c}^j(X') \to H_{L,c}^*(Y'_j, Y'_{j-1}) = H_c^*(Y'_j/L, Y'_{j-1}/L) \tag{4.6}$$

induced by the composition $Y' \to X' \to P$ is the identity for any j. Hence we also get an isomorphism in cohomology, so that $H^i(B_c^*(P))$ is a direct summand of $H^i(AB_{L,c}^*(X^K))$. This shows that condition (B) cannot hold. □

5. Relation to previous work

In this section we list several connections between our results and previous work. The assumptions on T-manifolds stated at the beginning of Section 3 remain in place. See Remark 4.3 for our use of Poincaré duality for the orbit space X/T.

5.1. **Cohomology of orbit spaces.** In [6, Cor. 2], Bredon proves that if $H_{T,c}^*(X)$ is free over R, then $H^i(X/T)$ vanishes for $i > \dim X - 2r$. (Bredon allows more general T-manifolds than we do here.) Observing that compared to [6] we have reversed the roles of compact and closed supports, we can extend his result as follows.

Corollary 5.1. *If $H_T^*(X)$ is a j-th syzygy, then $H_c^i(X/T) = 0$ for $i > \dim X - r - j$.*

Proof. On the E_2 page, the p-th column of the spectral sequence (3.7) converging to $H_c^*(X/T)$ is $H^p(B_c^*(X/T))$, and these graded vector spaces are concentrated in degrees $\le \dim X - 2r$. Since they are zero for $p > r - j$ by Theorem 4.4, $H_c^*(X/T)$ must vanishes in degrees greater than $\dim X - r - j$. □

5.2. **Toric varieties.** Let Σ be a regular fan in $\mathbb{Q}^r$ and X_Σ the associated smooth toric variety. There is an inclusion-reversing bijection between the faces of X_Σ/T and the cones $\sigma \in \Sigma$; the face $P_\sigma \in \mathcal{P}$ corresponding to a cone $\sigma \in \Sigma$ is of dimension $\dim P_\sigma = \operatorname{codim} \sigma$. The link L_σ of a cone $\sigma \in \Sigma$ is empty or of dimension $\operatorname{codim} \sigma - 1$. The homology of L_σ is related to the cohomology of P_σ by

$$H^i(B_c^*(P_\sigma)) \cong \tilde{H}_{\operatorname{codim} \sigma - i - 1}(L_\sigma). \tag{5.1}$$

(Both sides are concentrated in degree 0. Also recall that $\tilde{H}_{-1}(\emptyset) = \Bbbk$.) Hence one can reformulate the criterion given in Theorem 4.4 in terms of the links of the fan:

Corollary 5.2. *Let Σ be a regular fan in $\mathbb{Q}^r$ and X_Σ the associated smooth toric variety. Then $H_T^*(X_\Sigma)$ is a j-th syzygy if and only if $\tilde{H}_i(L_\sigma) = 0$ for all $\sigma \in \Sigma$ and all $-1 \le i < \min(j, \operatorname{codim} \sigma - 1)$.*

For the freeness of $H_T^*(X_\Sigma)$ this reduces to Reisner's criterion, *cf.* [7, Cor. 5.3.9], see also Barthel–Brasselet–Fieseler–Kaup [3, Lemma 4.5].

By Remark 3.6 we have for any $i \ge 0$ an isomorphism of graded algebras

$$H^i(AB_c^*(X_\Sigma)) = \bigoplus_{P \in \mathcal{P}} H^i(B_c^*(P)) \otimes R_P\, t_P. \tag{5.2}$$

Additively, such a decomposition has been obtained by Barthel–Brasselet–Fieseler–Kaup for the equivariant intersection cohomology of a possibly singular toric variety [3, eq. (3.4.2), Rem. 3.5]. The decomposition obtained by Bifet–De Concini–Procesi [4, Prop. 16] applies to toric varieties as well.

5.3. **Ext modules of Stanley–Reisner rings.** By a result of Bifet–De Concini–Procesi [4, Thm. 8], $H_T^*(X_\Sigma)$ is isomorphic to the Stanley–Reisner ring of Σ,

$$H_T^*(X_\Sigma) = \Bbbk[\Sigma]. \tag{5.3}$$

Using [1, Thm. 6.6], equivariant Poincaré duality [1, Prop. 3.5] and the identities (5.3) and (5.1), we can rewrite (5.2) as an isomorphism of graded vector spaces

$$\operatorname{Ext}_R^i(\Bbbk[\Sigma], R) \cong \bigoplus_{\sigma \in \Sigma} \tilde{H}_{\operatorname{codim} \sigma - i - 1}(L_\sigma) \otimes R_P\, t_P[-2r], \tag{5.4}$$

where "$[-2r]$" denotes a degree shift by $-2r$.

In the case where X_Σ is a toric subvariety of $\mathbb{C}^r$, so that $\Bbbk[\Sigma]$ is a quotient of R, such a decomposition has been obtained by Yanagawa [18, Prop. 3.1] and in an equivalent form by Mustaţă [17, Cor. 2.2].

5.4. **Torus manifolds.** In this section we additionally assume $\dim X = 2r$. Such an X is called a torus manifold if it is compact and $X^T \ne \emptyset$, *cf.* [16]. An open torus manifold is a T-manifold of the form $X = Y \backslash \pi^{-1}(S)$ where Y is a torus manifold and S a (possibly empty) union of faces of X/T, *cf.* [15]. Toric varieties are examples of open torus manifolds, and they are torus manifolds if they are complete. Note that an (open) torus manifold X has only finitely many fixed points, which correspond to the vertices (= rank 0 faces) of X/T. As a consequence, $H_T^*(X)$ is free over R if and only if $H^*(X)$ vanishes in odd degrees.

For a locally standard torus manifold X, Masuda–Panov proved that $H_T^*(X)$ is free over R if and only if every face of X/T is acyclic [16, Thm. 9.3]. Masuda then generalized this characterization to open torus manifolds [15, Thm. 4.3]. Using Theorem 4.4 we can recover his result and sharpen it. Unlike [16] and [15], we work

over a field of characteristic 0 instead of the integers, so we can drop the assumption that X be locally standard.

Proposition 5.3. *Assume* $\dim X = 2r$.

(1) $H_T^*(X)$ *is torsion-free if and only if and every face* $P \in \mathcal{P}$ *is acyclic and contains a vertex.*
(2) $H^*(X)$ *is free over* R *if and only if every face* $P \in \mathcal{P}$ *is acyclic and* ∂P *is acyclic for every non-compact face* $P \in \mathcal{P}$.

Note that the empty set is not acyclic since $\tilde{H}_{-1}(\emptyset) = \Bbbk$. Hence the condition on the boundaries in (2) implies that every face contains a vertex. If every face $P \in \mathcal{P}$ is acyclic, then X is called face-acyclic. That the freeness of $H_T^*(X)$ implies the face-acyclicity of X was first established by Bredon [6, Cor. 3].

Proof. Let P be a face of X/T, say of dimension i. Since X^P is an orientable manifold, so is $\dot{P}$. By Poincaré–Lefschetz duality, it follows that $H_c^{*+i}(\dot{P}) \cong H_*(P)$ is concentrated in degree 0 if and only if P is acyclic.

(1) Assume that there are non-acyclic faces, and let P be such a face of minimal dimension $i > 0$. Then $B_c^i(P)$ is the only column in the E_1 page of the spectral sequence (3.7) that does not vanish in negative degrees. Hence $H^i(B_c^*(P)) \neq 0$.

If all P are acyclic, then the spectral sequence collapses at the E_2 page. If in addition any face P of dimension $i > 0$ contains a vertex, then it must have at least one facet. Hence the differential $B_c^{i-1}(P) \to B_c^i(P)$, is non-trivial (*cf.* [14, Lemma 11.8]) and therefore $H^i(B_c^*(P)) = 0$. Otherwise, let P be a minimal face not containing a vertex, again of dimension $i > 0$. Then $H^i(B_c^*(P)) \neq 0$. In either case we conclude by invoking Theorem 4.4.

(2) Let P be a non-compact face of dimension $i > 0$. Using Poincaré duality for the boundary ∂P and the preceding discussion, we get

$$H^j(B_c^*(P)) \cong \begin{cases} \tilde{H}_{i-1-j}(\partial P) & \text{if } 0 < j < i, \\ 0 & \text{if } j = i. \end{cases} \tag{5.5}$$

Hence $H^j(B_c^*(P)) = 0$ for all $j > 0$ if and only if ∂P is acyclic. If P is compact and acyclic, then the $E_2 = E_\infty$ page of this spectral sequence is concentrated in the column $j = 0$. We again conclude with Theorem 4.4. □

Corollary 5.4. *Assume that* X *is compact and* $\dim X = 2r$. *Then* $H_T^*(X)$ *is torsion-free if and only if it is free.*

Proof. Every face of X/T is compact and contains a vertex. □

All "only if" parts in Proposition 5.3 and Corollary 5.4 do need the dimension condition on X. This is clear for Proposition 5.3 (2); for the other statements see the examples in Section 6.

Remark 5.5. Let X be a face-acyclic torus manifold. From Remark 3.6 we obtain Masuda–Panov's description of $H_T^*(X) = H^0(AB_c^*(X))$ as the face ring of the poset $\mathcal{P}$ [16, Cor. 7.6],

$$H_T^*(X) \cong \Bbbk\big[t_P : P \in \mathcal{P}\big] \;/\; \big(t_P t_Q - t_{P \vee Q} \sum_{O \in P \cap Q} t_O : P, Q \in \mathcal{P}\big). \tag{5.6}$$

This includes the formulas of Bifet–De Concini–Procesi (in the complete case) and of Davis–Januszkiewicz [10, Thm. 4.8].

5.5. **Equivariant injectivity.** Goertsches–Rollenske [12, Sec. 5] introduced the notion of equivariant injectivity for a compact T-manifold X and gave two characterizations of it. If every face $P \in \mathcal{P}$ contains a component of X^T, then this notion means that the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective or, equivalently, that $H_T^*(X)$ is torsion-free. For the manifolds we consider, a slight modification of their result [12, Thm. 5.9] is as follows:

Proposition 5.6. *The following are equivalent:*

(1) $H_T^*(X)$ *is torsion-free.*
(2) $\operatorname{depth}_{H^*(BL)} H_L^*(X^K) \geq 1$ *for every subtorus* $K \subset T$ *with quotient* $L = T/K$.
(3) *The restriction map*

$$H^*(P) \to \bigoplus_{Q <_1 P} H^*(Q).$$

is injective for every face $P \in \mathcal{P}$ *of codimension* $< r$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a special case of Proposition 2.1. That these conditions are equivalent to (3) is a consequence of Theorem 4.4. This can be seen as follows:

Let $Q <_1 P \in \mathcal{P}$, $i = \operatorname{rank} P$ and $M = \dot{P} \cup \dot{Q}$. By Poincaré–Lefschetz duality for the "orbifold with boundary" M one has a commutative diagram

$$\begin{tikzcd} H_c^{*-1}(\dot{Q}) \arrow[r, "\delta"] \arrow[d, "\cap\,\partial\mu"'] & H_c^*(M, \dot{Q}) \arrow[d, "\cap\,\mu"] \\ H_*(\dot{Q}) \arrow[r] & H_*(M), \end{tikzcd} \tag{5.7}$$

whose vertical arrows are isomorphisms, *cf.* [14, §11.4]. Here $\mu \in H_*^c(\dot{P})$ is an orientation of $\dot{P}$ and $\partial\mu \in H_*^c(\dot{Q})$ the induced orientation of $\dot{Q}$. Since $H_*(M) = H_*(P)$ and $H_*(\dot{Q}) = H_*(Q)$, we conclude that the differential

$$B_c^{i-1}(P) = \bigoplus_{Q <_1 P} H_c^{*-1}(\dot{Q}) \longrightarrow H_c^*(\dot{P}) = B_c^i(P) \tag{5.8}$$

is surjective if and only if the combined restriction map

$$H^*(P) \to \bigoplus_{Q <_1 P} H^*(Q). \tag{5.9}$$

is injective. $\square$

Remark 5.7. Let $Q <_1 P$, and let U be a "collar" of $\dot{Q}$ in $\dot{P} \cup \dot{Q}$ (obtained from an invariant tubular neighbourhood of $X^{\dot{Q}}$ in $X^{\dot{P}} \cup X^{\dot{Q}}$). Then the restriction map for this pair of faces factorizes as

$$H^*(P) \xrightarrow{\cong} H^*(\dot{P}) \longrightarrow H^*(U \setminus \dot{Q}) \xleftarrow{\cong} H^*(U) \xrightarrow{\cong} H^*(\dot{Q}) \xleftarrow{\cong} H^*(Q). \tag{5.10}$$

Hence one can equivalently look at the maps $H^*(\dot{P}) \to H^*(U \setminus \dot{Q})$, and this is what Goertsches–Rollenske actually do. (Note that we are implicitly using our restriction on the dimension of the strata. Goertsches–Rollenske do not need this assumption.)

6. Examples

6.1. **Non-compact examples.** Let Y be a face-acyclic torus manifold of dimension $2r$. Choose two fixed points $y_1 \neq y_2$ of Y and consider the open torus manifold $X = Y \setminus \{y_1, y_2\}$.

Let $P = y_1 \vee y_2$ be the minimal face of Y/T containing both y_1 and y_2. Then

$$H^i(B_c^*(Q)) \cong \begin{cases} \Bbbk & \text{if } Q \subset P \text{ and } i = 1, \\ 0 & \text{otherwise if } i > 0. \end{cases} \tag{6.1}$$

By Theorem 4.4, $H_T^*(X)$ is a $(p-1)$-st, but not a p-th syzygy, where $p = \dim P$.

The "smallest" example X such that $H_T^*(X)$ is an $(r-1)$-st syzygy, but not free is obtained by taking $Y = (\mathbb{CP}^1)^r = (S^2)^r$, so that Y/T is an r-dimensional cube, and removing two fixed points corresponding to opposite vertices. For this space $H_T^*(X)$ has been explicitly computed in [1, Sec. 7.2]:

$$H_T^*(X) \cong \bigoplus_{i=0}^{r-2} R[2i]^{\binom{r}{i}} \oplus K_{r-1}[2(r-1)], \tag{6.2}$$

where K_{r-1} is the $(r-1)$-st syzygy in the Koszul resolution of R, *cf.* [1, Sec. 2.4].

6.2. **Mutants.** Let X be the 7-dimensional mutant constructed in [11]. We apply Theorem 4.4 to verify that $H_T^*(X)$ is torsion-free, but not reflexive (hence not free).

The mutant comes equipped with a smooth action of $T = (S^1)^3$, and X/T is topologically a 4-ball. Consider the standard T-action on $\mathbb{C}^3$ and its compactification $Y \cong S^6$. Then Y/T is topologically a 3-ball. Removing the centres $p \in X/T$ and $q \in Y/T$, we obtain the Hopf fibration

$$(X/T) \setminus \{p\} \to (Y/T) \setminus \{q\}. \tag{6.3}$$

(One can use this to construct X/T and X.)

The isotropy groups occurring in X and Y are the coordinate subtori of T. For such a subtorus K of rank $0 < k < r$, one has $Y^K = S^{2(r-k)}$, the compactification of $\mathbb{C}^{2(r-k)}$, and $X^K \cong S^{2(r-k)} \times S^1$, where T/K acts trivially on S^1. Hence $H^i(B_c^*(P)) = 0$ for $P = X^K/T$ and $i > 0$. It is therefore enough to look at $Q = X/T$ itself. In this case the complex (3.7) has the following form:

$$B_c^*(Q)\colon \begin{array}{c|cccc} 1 & \Bbbk^2 & \Bbbk^3 & \Bbbk^3 & \Bbbk \\ 0 & \Bbbk^2 & \Bbbk^3 & \Bbbk^3 & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array} \qquad H^*(B_c^*(Q))\colon \begin{array}{c|cccc} 1 & \Bbbk & 0 & 0 & 0 \\ 0 & \Bbbk & 0 & \Bbbk & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array} \tag{6.4}$$

(The zeroth row computes $H^*(S^2)$ because $H_c^*(\dot{Q})$ is concentrated in top degree; the first row then follows from the fact that the spectral sequence converges to $H^*(Q) = \Bbbk$.) Hence $H_T^*(X)$ is a first, but not a second syzygy.

We can also compute $H_T^*(X)$ independently of [11, Sec. 5]. From Proposition 3.5 and Remark 3.6 we get isomorphisms of R-modules

$$H^i(AB^*(X)) \cong \begin{cases} R \oplus R[1] \oplus R[6] \oplus R[7] & \text{if } i = 0, \\ \Bbbk & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

By [1, Lemma 5.5] there must be a higher differential, whence

$$H_T^*(X) \cong R \oplus \mathfrak{m}[1] \oplus R[6] \oplus R[7]. \tag{6.6}$$

Remark 6.1. As part of the calculation we showed $H^i(B_c^*(P)) = 0$ for any proper face $P < X/T$ and any $i > 0$. In other words, $H_L^*(X^K)$ is free over $H^*(BL)$ for any non-trivial coordinate subtorus $K \subset T$ with quotient $L = T/K$.

This is not a coincidence. Since $H_T^*(X)$ is a first syzygy, so is $H_L^*(X^K)$ for any subtorus $K \subset T$ by Corollary 2.2. If K is non-trivial, then L is of rank at most 2. Because X^K is again a Poincaré duality space, it follows from [1, Prop. 5.10] that $H_L^*(X^K)$ is free over $H^*(BL)$.

Remark 6.2. Let $r - 1 \in \{2, 4, 8\}$. A computation analogous to the one above shows that instead of S^6 one can start with any face-acyclic $2r$-dimensional torus manifold Y whose orbit space is homeomorphic to an r-ball. The equivariant cohomology of the T-manifold X obtained by means of the Hopf fibration $S^{2r-3} \to S^{r-1}$ in the same way as above is then torsion-free, but not reflexive. The cases $Y = S^{2r}$ considered in [11] were the only examples known so far.

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